14. Ains, E.L., Ordinary Differential Equations. Khar'kov, DNTVU, 1939.
15. Barnes, E.W., The asymptotic expansion of integral functions defined by generalized hypergeometric series. Proc., London, Math. Soc., Vol. 2, No. 5, 1907.
16. Whittaker, E.T. and Watson, N.G., Course of Modern Analysis, p. 2, (Russian Translation) Fizmatgiz, 1963.

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ON THE DETERMINATION OF KINETIC STRESS FUNCTIONS IN ELASTODYNAMICS PROBLEMS

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The purpose of the paper is the development of a new method of solving dynamic problems of elasticity theory by introducing kinetic stress functions [1 to 3]. Equations which the kinetic stress functions satisfy are presented here, and the form of the general solution of these equations is found.

Let us consider the square of a line element in some Riemann space, which we shall designate as generating:

$$ds^2 = [1 + \varepsilon\varphi_{kk}(x^j, t)]\, dx^k dx^k - c^2 [1 + \varepsilon\varphi_4(x^j,t)]\, dt^2 \quad (k,\ j = 1,\ 2,\ 3) \tag{1}$$

where ε is an arbitrary small parameterm c^2 a constant to be determined, $\varphi_{kk}(x^j, t) = \varphi_k(x^1, x^2, x^3, t)$ the kinetic stress functions. It is seen from (1) that for $\varepsilon = 0$ the Riemann space degenerates into a Euclidean space. We assume that this Euclidean space contains the continuum being studied. Functional derivatives of the components of the fundamental metric tensor of the generating Riemann space define the kinetic stress tensor as $\varepsilon \to 0$.

We assume that the energy momentum tensor is proportional to the functional derivative of the fundamental geometric invariant [4]. Let us set

$$T^{\mu\nu} = \varepsilon^{-1}(R^{\mu\nu} - {}^1/_2\, g^{\mu\nu} R) \tag{2}$$

where $T^{\mu\nu}$ is the energy-momentum tensor; the remaining notation is standard.

As a result of passing to the limit as $\varepsilon \to 0$ we obtain a general solution of the equations of motion of a continuum element from (2) [2]:

$$\sigma_{ii} - \rho v_{ii}^2 = \frac{1}{2}\left[\frac{\partial^2(\varphi_j + \varphi_4)}{\partial x^k \partial x^k} + \frac{\partial^2(\varphi_k + \varphi_4)}{\partial x^j \partial x^j} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\varphi_k + \varphi_j)\right] \tag{3}$$

$$\sigma_{kj} - \rho v_k v_j = -\frac{1}{2}\frac{\partial^2(\varphi_i + \varphi_4)}{\partial x^k \partial x^k} \tag{4}$$

$$\rho v^i = -\frac{1}{2c^2}\frac{\partial^2(\varphi_k + \varphi)}{\partial x^i \partial t} \tag{5}$$

$$\rho = \frac{1}{2c^2}\left[\frac{\partial^2(\varphi_3 + \varphi_2)}{\partial x^1 \partial x^1} + \frac{\partial^2(\varphi_1 + \varphi_3)}{\partial x^2 \partial x^2} + \frac{\partial^2(\varphi_1 + \varphi_2)}{\partial x^3 \partial x^3}\right] \tag{6}$$

Here σ_{ik} is the stress tensor, **v** the velocity of a continuum element, ρ the density. The indices i, k, j generate a cyclic permutation of the numbers 1, 2, 3. We henceforth neglect nonlinear terms in the components of the three-dimensional portion of the kinetic stress tensor in Expressions (3) to (6). The generality of (3) to (6) results, in particular, from the possibility of introducing an arbitrary orthogonal curvilinear coordinate system in the Euclidean space.

In the case of a compressible medium the density is approximately representable by the relationship

$$\rho = \rho_0 (1 - \operatorname{div} \mathbf{u}), \qquad \rho_0 = \text{const} \tag{7}$$

In combination with (7), the equality (5) leads to such values of the linear portion of the components of the velocity vector:

$$v^i = -\frac{1}{2\rho_0 c^2} \frac{\partial^2 \Phi_i}{\partial x^i \partial t} \qquad (i = 1, 2, 3; \quad \Phi_i = \varphi_j + \varphi_k) \tag{8}$$

The fundamental substitutions (3) to (6) are independent of the equations of state, i.e., they are applicable to a study of the dynamics of elastic and plastic solids, as well as to a viscous fluid. Now, let us consider elastodynamics as a particular case. In this case the equations of state agree with the generalized Hooke's law.

After elimination of nonlinear terms and obvious transformations, we find the following equations from Hooke's law and relationships (3), (4), (8):

$$2\varphi_4 - \varphi_i + \frac{\rho_0 c^2 - 2\mu}{\rho_0 c^2} \Phi_j + \frac{\rho_0 c^2 - 2\mu}{\rho_0 c^2} \Phi_k = \alpha_j (x^1, x^2, x^3, t) \tag{9}$$

$$\nabla^2 \Phi_i - \frac{\rho_0}{\mu} \frac{\partial^2 \Phi_i}{\partial t^2} = -\frac{\lambda + \mu}{\mu} \theta_1 + \frac{c^2}{2\mu} \left[4f_{ii} - \frac{\partial^2 \alpha_k}{\partial x^k \partial x^k} - \frac{\partial^2 \alpha_j}{\partial x^j \partial x^j} \right] \tag{10}$$

$$\left(\theta_1 = \frac{\partial^2 \Phi_1}{\partial x^1 \partial x^1} + \frac{\partial^2 \Phi_2}{\partial x^2 \partial x^2} + \frac{\partial^2 \Phi_3}{\partial x^3 \partial x^3} \right)$$

Here the α_j are undetermined functions of integration. Eqs. (6) and (7) reduce to the relationship

$$\rho_0 (1 - \operatorname{div} \mathbf{u}) = \frac{1}{2c^2} \theta_1 \tag{11}$$

Let us recall that

$$\operatorname{div} \mathbf{u} = \frac{1}{3k} (\sigma_{11} + \sigma_{22} + \sigma_{33}) \tag{12}$$

Taking account of (12) and (3), we obtain from (11)

$$\nabla^2 \varphi_4 - \frac{1}{2c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{3\lambda + 2\mu + \rho_0 c^2}{2\rho_0 c^2} \theta_1 + 3\lambda + 2\mu \qquad (\Phi = \Phi_1 + \Phi_2 + \Phi_3) \tag{13}$$

Therefore, the functions $\Phi_k (x^j, t)$ and $\varphi_4 (x^j, t)$ should satisfy the system of Eqs. (9), (10) and (13). These equations contain an undetermined constant c^2 and the undetermined functions of integration. To determine c^2 we form (10) by a different method. We find from (8)

$$u^i = -\frac{1}{2\rho_0 c^2} \frac{\partial \Phi_i}{\partial x^i} + \frac{\partial \gamma_i}{\partial x^i} \tag{14}$$

Here $\partial \gamma_i / \partial x^i$ are integration functions. Substituting (14) into the generalized Hooke's law, and equating the normal stresses to (3), we again find

$$\nabla^2 \Phi_i - \frac{1}{c^2} \frac{\partial^2 \Phi_i}{\partial t^2} = -\frac{\lambda + 2\mu - \rho_0 c^2}{\rho_0 c^2} \theta_1 + \left[(2\lambda + 4\mu) + \frac{\theta_2}{2} - \frac{1}{2} \frac{\partial^2 \beta_i}{\partial x^i \partial x^i} \right] \tag{15}$$

$$\theta_2 = \frac{\partial^2 \beta_1}{\partial x^1 \partial x^1} + \frac{\partial^2 \beta_2}{\partial x^2 \partial x^2} + \frac{\partial^2 \beta_3}{\partial x^3 \partial x^3} \tag{16}$$

Here the β_i are arbitrary integration functions. Eqs. (10) and (15) should be identical, as definitions of the same physical process. Comparing these equations we find

$$c^2 = \mu / \rho_0 \tag{17}$$

Hence, the expressions in the square brackets in the right-hand sides of (10) and (15) will be equal. As we shall show below, the generality of the obtained equations is not disturbed if particular expressions of the arbitrary integration functions are selected so as to eliminate them from (10) and (15). We then find

$$\left(\nabla^2 - \frac{\rho_0}{\mu}\frac{\partial^2}{\partial t^2}\right)\Phi_i = -\frac{\lambda+\mu}{\mu}\theta_1 \tag{18}$$

For the value found for c^2 the system (9) reduces to Eq. of the form

$$\Phi_1 + \Phi_2 + \Phi_3 = 2\varphi_4 - (\lambda + {}^2/_3\,\mu)\,\delta_{km}x^kx^m \tag{19}$$

It is easy to note that (13), defining φ_4, is a consequence of (18) with (17) and (19). Moreover, setting

$$\Phi_i = \Psi_i + \mu/3\,(x^i)^2 + a\,[(x^k)^2 + (x^j)^2] + bt^2 \tag{20}$$

and determining the coefficients a and b from the conditions of eliminating the homogeneous field of static stresses, we finally obtain

$$\sigma_{ii} = \frac{1}{2}\left\{\frac{\partial^2(\Psi_i+\Psi_k)}{\partial x^k\partial x^k} + \frac{\partial^2(\Psi_j+\Psi_i)}{\partial x^j\partial x^j} - \frac{\rho_0}{\mu}\frac{\partial^2\Psi_i}{\partial t^2}\right\} \tag{21}$$

$$\sigma_{jk} = -\frac{1}{2}\frac{\partial^2(\Psi_j+\Psi_k)}{\partial x^j\partial x^k}, \qquad u_i = -\frac{1}{2\mu}\frac{\partial\Psi_i}{\partial x^i} \tag{22}$$

Here the Ψ_i are new kinetic stress functions. It follows from (20) and (18) that the kinetic stress functions Ψ_i are the solutions of a system of three differential Eqs.

$$\Box_2\Psi_i = -\frac{\lambda+\mu}{\mu}\Delta_1 \quad \left(\Box_2 = \nabla^2 - \frac{\rho_0}{\mu}\frac{\partial^2}{\partial t^2}, \quad \Delta_1 = \frac{\partial^2\Psi_1}{\partial x^1\partial x^1} + \frac{\partial^2\Psi_2}{\partial x^2\partial x^2} + \frac{\partial^2\Psi_3}{\partial x^3\partial x^3}\right) \tag{23}$$

Thus the solution of elastodynamics problems reduces to solving the system (23) in conjunction with relationships (21) and (22), from which the boundary and initial conditions for the functions Ψ_i result. The generality of the obtained solution is confirmed by the fact that we obtain the homogeneous Lamé equations by differentiating (23) and substituting the relationships (22).

Conversely, we again find (23) from the Lamé equations and (22) having integrated with respect to x^i ($i = 1, 2, 3$). The undetermined integration functions manifested here do not affect the displacement and stress field. We omit the elementary proof of this, almost obvious, fact.

The system (23) differs from the Lamé equations in that the right-hand sides of these equations are identical, which permits expression of two desired functions in terms of the third

$$\Psi_1 = \Psi_3 + F_1, \qquad \Psi_2 = \Psi_3 + F_2 \tag{24}$$

The functions F_1 and F_2 are solutions of Eq.

$$\Box_2 F_i = 0 \qquad (i = 1, 2) \tag{25}$$

After substituting (24) into the third Eq. of (23) we find

$$\Box_1\Psi_3 = -\frac{\lambda+\mu}{\lambda+2\mu}\left(\frac{\partial^2F_1}{\partial x^1\partial x^1} + \frac{\partial^2F_2}{\partial x^2\partial x^2}\right) \qquad \left(\Box_1 = \nabla^2 - \frac{\rho_0}{\lambda+2\mu}\frac{\partial^2}{\partial t^2}\right) \tag{26}$$

Hence, there results

$$\Box_2\Box_1\Psi_3 = 0 \tag{27}$$

Eq. (27) agrees with one of the equations mentioned by A.I. Lur'e [5] in finding the general solution of the linear elastodynamics equations.

As is seen from (25), (27) and (22), the solution of elastodynamics problems for displacements given on the body surface admits of an autonomous determination of the functions Ψ_i. These functions are connected by boundary conditions in other cases.

In conclusion, let us recall that the formulation of the type (3) to (6), permits extension of the expounded method to other cases of continuum dynamics, as well as to nonlinear problems. In this case, in place of the stresses σ_{ik}, the kinetic stresses should be considered without neglecting the nonlinear terms in (3) to (6).

BIBLIOGRAPHY

1. Kil'chevskii, N.A., On stress, velocity and density functions in static and dynamic problems of continuum mechanics. Dokl. Akad. Nauk SSSR, Vol. 92, No. 5, 1953.

2. Kil'chevskii, N.A., Elements of Tensor Calculus and its Applications to Mechanics. Moscow, Gostekhizdat, 1954.
3. Kil'chevskii, N.A., Trudy II Vses. s'ezda po teoret. i prikl. mekhan., Moscow, "Nauka" Press, Vol. I, 1965.
4. Eddington, A.S., Theory of Relativity, Moscow-Leningrad, Gostekhizdat, 1934.
5. Lur'e, A.I., On the theory of systems of linear differential equations with constant coefficients. Trudy Leningr. Industrial'n. Inst., No. 6, Issue 3, 1937.

Translated by M.D.F.

ASYMPTOTIC SOLUTION OF A CLASS OF INTEGRAL EQUATIONS AND ITS APPLICATION TO CONTACT PROBLEMS FOR CYLINDRICAL ELASTIC BODIES

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A special class of integral equations of the first kind with irregular difference kernel of complex structure dependent on a nondimensional parameter λ is considered. The asymptotic solution of this integral equation is constructed for large values of λ as a double series in powers of λ^{-1} and $\ln \lambda$.

The obtained results are utilized to study axisymmetric problems of the interaction between a stiff belt and the surface of an infinite elastic cylinder, as well as the interaction between a stiff bushing and the surface of an infinite cylindrical cavity in elastic space.

Finally, under the customary assumptions of Hertz theory, the problem of interaction between an elastic belt and infinite elastic cylinder is examined on the basis of the solution of the first two problems.

1. **Investigation of the structure of the solution of the integral equation and construction of the asymptotic solution for large values of the parameter λ.** Let us consider an integral Eq. of the form

$$\int_{-1}^{1} \left\{-\ln\frac{|x-t|}{\lambda} + a_{20}\frac{|x-t|}{\lambda} + a_{30} + F\left(\frac{x-t}{\lambda}\right)\right\}\varphi(t)\,dt = \pi f(x) \qquad (|x| \leqslant 1) \tag{1.1}$$

$$F(y) = \ln|y|\,F_1(y) + |y|\,F_2(y) + F_3(y) \tag{1.2}$$

The functions $F_i(y)$ will be continuous with all their derivatives for all values $-2/\lambda \leqslant y = (x-t)/\lambda \leqslant 2/\lambda$ and will behave as $O(y^2)$ for $y \to 0$.

Hence it follows that the function $F(y) \in H_1^{\alpha}(-1, 1)$, $0 < \alpha < 1$ where $H_n^{\alpha}(-\beta, \beta)$ denotes the space of functions whose n-th derivative satisfies the Hölder condition with exponent α for $|x| \leqslant \beta$.

We shall moreover assume that $f(x) \in H_p^{\alpha}(-1, 1)$, $\alpha > 0$, $p \geqslant 1$.

Following [1], let us represent (1.1) as an equivalent integral equation of the second kind

$$\omega(x) = \frac{P}{\pi} - \frac{1}{\pi}\int_{-1}^{1}\frac{f'(t)\sqrt{1-t^2}}{t-x}\,dt + \frac{1}{\pi^2}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\,dt\int_{-1}^{1}\left\{a_{20}\,\mathrm{sgn}\,(t-y) + \right.$$